# Qubitek gazdag dinamikája

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CTU CZECH TECHNICAL UNIVERSITY IN PRAGUE

TECHNISCHE UNIVERSITÄT DARMSTADT

# Quantum theory: linear or nonlinear?

1., Closed systems - unitary operators - linear evolution

2., Quantum channels - completely positive maps - linear evolution

## If quantum states evolved nonlinearly

- hard problems (NP complete) easily solved (in polynomial time)
  D. S. Abrams and S. Lloyd, PRL 81, 3992 (1998)
- e.g. search using the Gross-Pitaevskii equation
  D. A. Meyer and T. G. Wong, New J. Phys. 15, 063014 (2013)
- quick discrimination of nonorthogonal states - generic feature
  A. M. Childs and J. Young, Phys. Rev. A, 93, 022314 (2016)

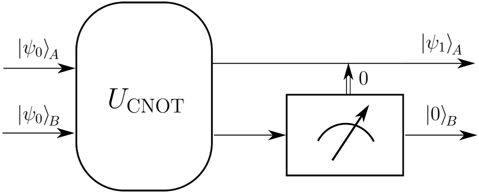
# Nonlinear transformations by selective evolution

3., Measurements

- ✓ projection (von Neumann)
- ✓ probabilistic (Born)
- ✓ information gained
- ☞ information feed-back
- ☞ post-selection

↯ breaking linearity ↯

# Transformation of a qubit




$$|\psi_0\rangle = \frac{1}{\sqrt{1+|z|^2}}\left(|0\rangle + z\,|1\rangle\right)$$

H. Bechmann-Pasquinucci et al. Phys. Lett. A **242**, 198 (1998).

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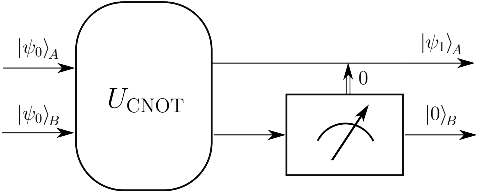


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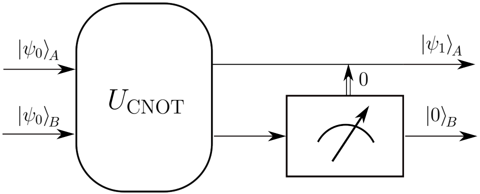


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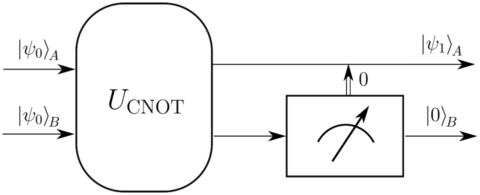


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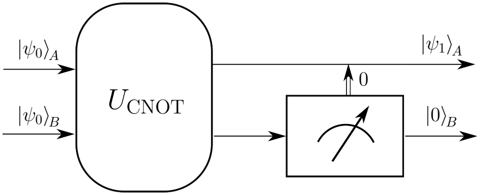
- after projecting qubit B to $|0\rangle$:

$$|\psi_1\rangle_A = \frac{1}{\sqrt{1+|z|^2}} \Big(|0\rangle + z^2\,|1\rangle\Big)$$

# Transformation of a qubit

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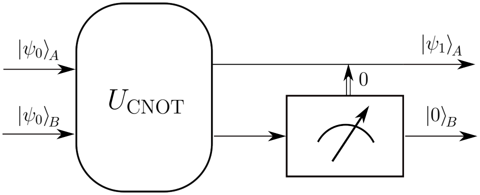
- after projecting qubit B to $|0\rangle$:

$$|\psi_1\rangle_A = \frac{1}{\sqrt{1+|z|^2}}\Big(|0\rangle + z^2\,|1\rangle\Big) \qquad \longrightarrow \qquad f(z) = z^2$$

# Transformation of a qubit

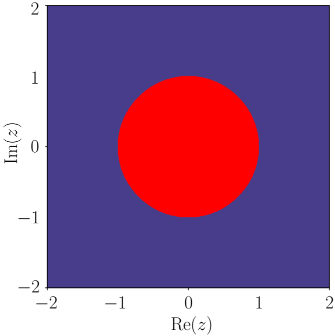



$$|\psi_0\rangle = \frac{1}{\sqrt{1+|z|^2}}\left(|0\rangle + z\,|1\rangle\right)$$

$$\downarrow$$

$$|\psi_1\rangle = \frac{1}{\sqrt{1+|z|^4}}\left(|0\rangle + z^2\,|1\rangle\right)$$

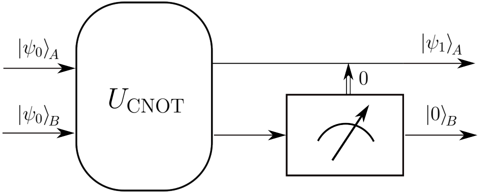
H. Bechmann-Pasquinucci et al. Phys. Lett. A **242**, 198 (1998).




Iteration of $f(z) = z^2$ (complex plane)

- $|z| < 1 \to 0$ (stable fixed point)
- $|z| > 1 \to \infty$ (stable fixed point)
- $|z| = 1 \to$ **no convergence**

# Transformation of a qubit

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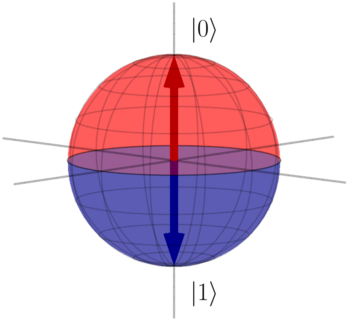



Iteration of $f(z) = z^2$
(Bloch sphere)

- ☞ $|z| < 1$ states converge to $|0\rangle$
- ☞ $|z| > 1$ states converge to $|1\rangle$
- ☞ $z = 1$ **weird points: the Julia set**

# Iterative nonlinear quantum protocols

- Ensemble of qubits in *pure state* $|\psi_0\rangle \sim |0\rangle + z\,|1\rangle \quad (z \in \mathbb{C})$
  1. Take them pairwise:
  $$|\Psi_0\rangle = |\psi_0\rangle_A \otimes |\psi_0\rangle_B$$
  2. Apply an **entangling** two-qubit operation $U$
  3. Measure the state of qubit $B$ — keep $A$ only for result $0$
- Smaller ensemble in *pure state* $|\psi_1\rangle \sim |0\rangle + f(z)\,|1\rangle$
- Quantum magnification bound: exponential downscaling of the ensemble

$$U \leftrightarrow f(z) = \frac{a_0 z^2 + a_1 z + a_2}{b_0 z^2 + b_1 z + b_2}$$

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016)

# Historical remarks on complex dynamics

- Iterated rational polynomials: $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \quad f^{\circ n} \to ?$
- One century of complex chaos:

  **1871** idea of iterated functions by Ernst Schröder

  *Ueber iterirte Functionen.*, Math. Ann.

  **1906** first weird example by P. Fatou: $z \mapsto z^2/(z^2+2)$

  **1920ies** G. Julia, S. Lattès, & . . .

  **1970ies** Computers help visualize: Mandelbrot & . . .

- A good book:

  J.W. Milnor *Dynamics in One Complex Variable*, (Vieweg, 2000)

# Iterative dynamics - examples

## CNOT gate plus a single qubit gate

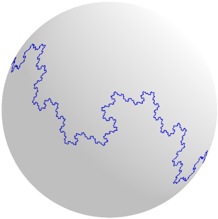
$$U = \begin{pmatrix} \cos\theta & \sin\theta\, e^{i\varphi} \\ -\sin\theta\, e^{-i\varphi} & \cos\theta \end{pmatrix}$$

## Family of maps over $\hat{\mathbb{C}}$:

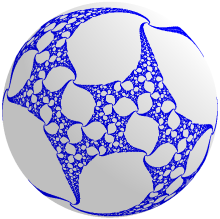
$$z \mapsto f_p(z) = \frac{z^2 + p}{1 - p^* z^2} \qquad p = \tan\theta\, e^{i\varphi}$$

$p \in \mathbb{C}$ parameter of the gate

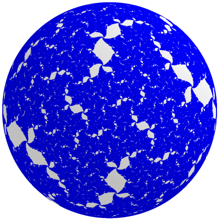
# Iterative dynamics - Julia sets on the Bloch sphere



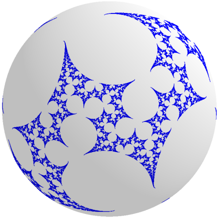
**(a)** $\theta = 0.4, \varphi = \frac{\pi}{2}$



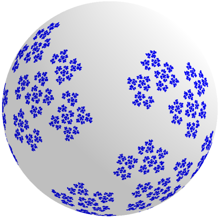
**(b)** $\theta = 0.55, \varphi = \frac{\pi}{2}$



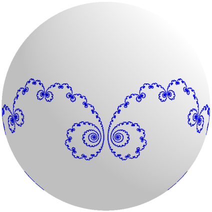
**(c)** $\theta = 0.633, \varphi = \frac{\pi}{2}$



**(d)** $\theta = 1.05, \varphi = \frac{\pi}{2}$



**(e)** $\theta = 0.5, \varphi = 0.5$

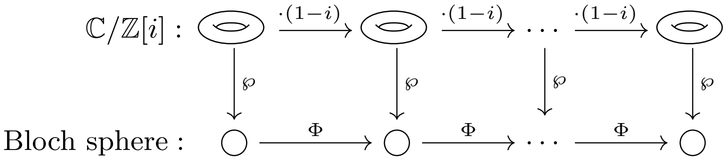


**(f)** $\theta = 0.232, \varphi = 0$

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016).

# Lattès map: $J = \hat{\mathbb{C}}$

$$f(z) = \frac{z^2 + i}{iz^2 + 1}, \quad p = i$$



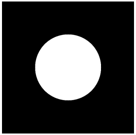

A commutative diagram:

- map on the Bloch sphere $\leftrightarrow \times(1 - i)^n$ on the torus
- all initial states are weird
- ergodicity

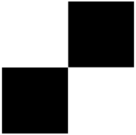
Lattès, S (1918), Les Comptes rendus de l'Académie des sciences, 166: 26-28

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016).

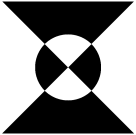
# Lattès map: ergodic dynamics



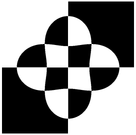
**(a)** $|z| > 1$



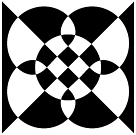
**(b)** $|f(z)| > 1$



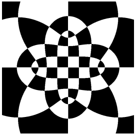
**(c)** $|f^{\circ 2}(z)| > 1$



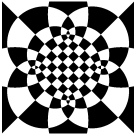
**(d)** $|f^{\circ 3}(z)| > 1$



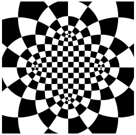
**(e)** $|f^{\circ 4}(z)| > 1$



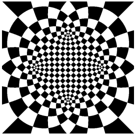
**(f)** $|f^{\circ 5}(z)| > 1$



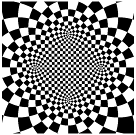
**(g)** $|f^{\circ 6}(z)| > 1$



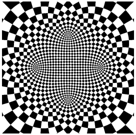
**(h)** $|f^{\circ 7}(z)| > 1$



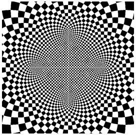
**(i)** $|f^{\circ 8}(z)| > 1$



**(j)** $|f^{\circ 9}(z)| > 1$



**(k)** $|f^{\circ 10}(z)| > 1$



**(l)** $|f^{\circ 11}(z)| > 1$

# Lattès map with noisy initial states

Dynamics represented by $\mathbb{R}^3 \to \mathbb{R}^3$ functions:

$$u' = \frac{u^2 - v^2}{1 + w^2}, \quad v' = \frac{2w}{1 + w^2}, \quad w' = -\frac{2uv}{1 + w^2}$$

No book by Milnor! :-(

Asymptotics: all mixed initial states → completely mixed state

O. Kálmán, T. Kiss and I. Jex, J Russ Laser Res 39: 382 (2018)

# CNOT + Hadamard gate: phase transition

Noisy (mixed) initial states:

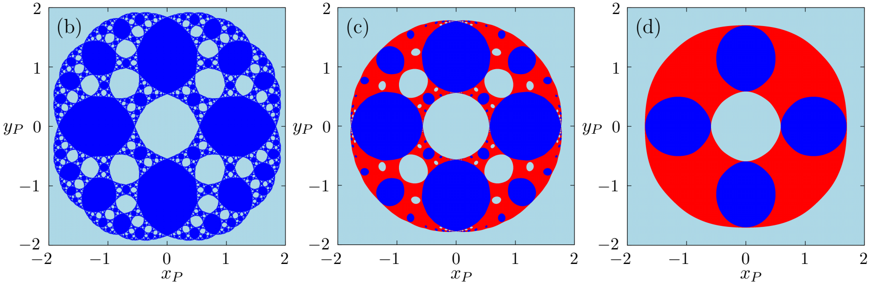
$$\rho \xrightarrow{\mathcal{M}} \rho' = U_H \, \frac{\rho \odot \rho}{\mathrm{Tr}(\rho \odot \rho)} \, U_H^\dagger$$

where

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \rho = \frac{1}{2} \begin{pmatrix} 1+w & u-iv \\ u+iv & 1-w \end{pmatrix}$$

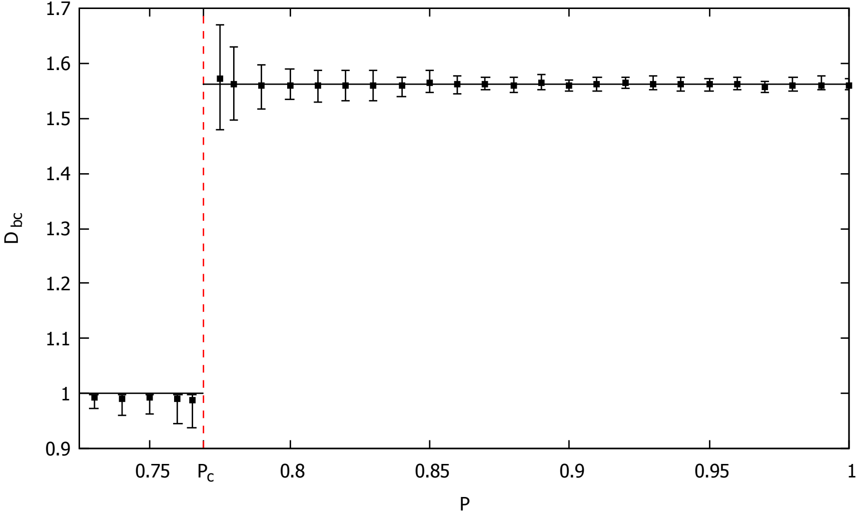
Purity: $P = \mathrm{Tr}(\rho^2) = (1 + u^2 + v^2 + w^2)/2 \leq 1$

# Convergence for different purities $P$



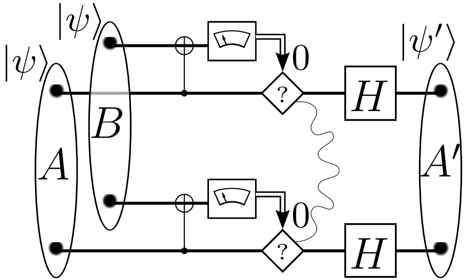

$P = 1$ $P = 0.87$ $P = 0.75$

# Fractal dimension $D_{bc}$ as a function of purity $P$




M. Malachov, I. Jex, O. Kálmán, and T. Kiss, Chaos 29, 033107 (2019)

# LOCC scheme with 2 qubits




$$|\psi\rangle = c_1|00\rangle + c_2|01\rangle + c_3|10\rangle + c_4|11\rangle$$

$$|\psi'\rangle = U_H \otimes U_H \left[\mathcal{N}(c_1^2|00\rangle + c_2^2|01\rangle + c_3^2|10\rangle + c_4^2|11\rangle)\right]$$

# 2 qubits: chaotic entanglement

## Asymptotic states

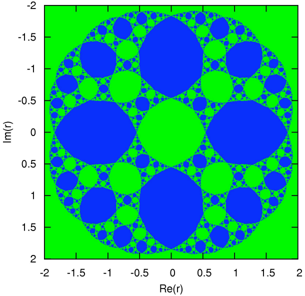
**Green:** Fully entangled:

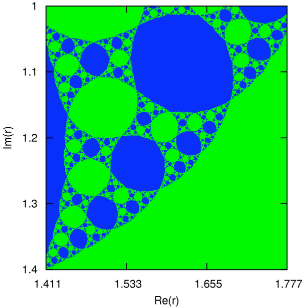
$$|\psi^{(\infty)}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

**Blue:** Completely separable, oscillatory:

$$|\psi^{(\infty)}\rangle \rightarrow \left\{|00\rangle\,,\ \frac{1}{2}\left(|0\rangle + |1\rangle\right)\otimes\left(|0\rangle + |1\rangle\right)\right\}$$

T. Kiss, S. Vymětal, L. D. Tóth, A. Gábris,
I. Jex, G. Alber, PRL **107**, 100501 (2011)



# An application for state orthogonalization

- nonlinear transformation: $f_{\varphi=0} = \dfrac{2z}{1+z^2}$

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

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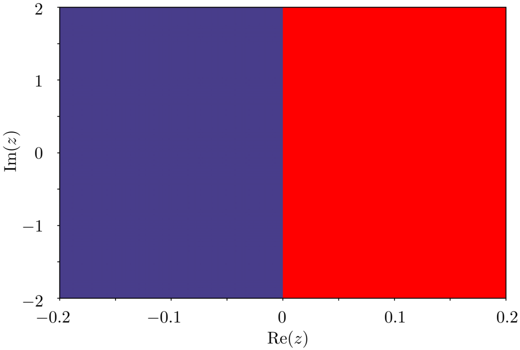


- two superattractive fixed points: $1$ and $-1$
- Julia set: imaginary axis

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

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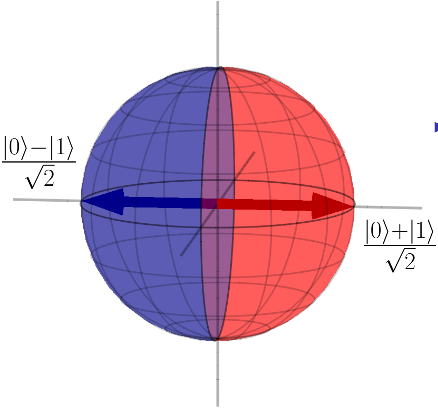


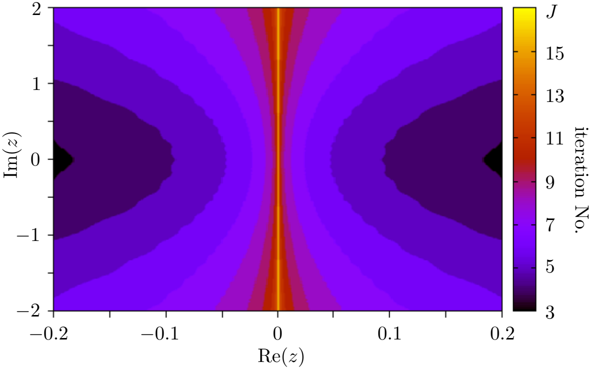

- Julia set: longitudinal great circle through $y$ axis
  - equally separates regions of convergence

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

# An application for state orthogonalization

- nonlinear transformation: $f_{\varphi=0} = \dfrac{2z}{1+z^2}$

- From highly overlapping to almost orthogonal in only 3 steps




$$|\Psi_0\rangle_I = \frac{|0\rangle + 0.2\,|1\rangle}{\sqrt{1+(0.2)^2}}$$

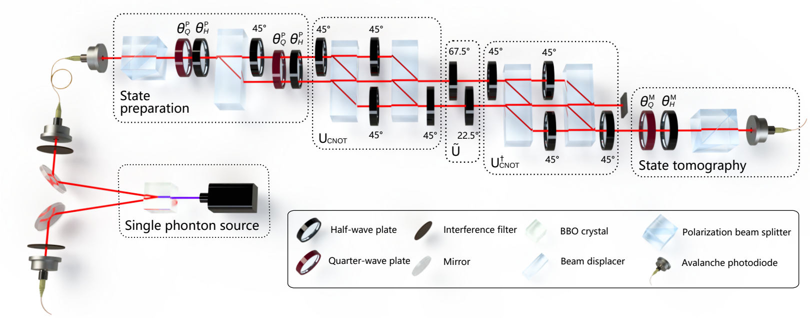
$$|\Psi_0\rangle_{II} = \frac{|0\rangle - 0.2\,|1\rangle}{\sqrt{1+(0.2)^2}}$$

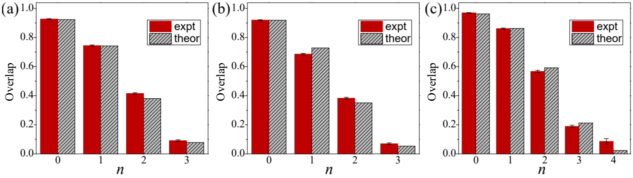
$$_I\langle\Psi_0\,|\,\Psi_0\rangle_{II} \approx 0.92$$

$$_I\langle\Psi_3\,|\,\Psi_3\rangle_{II} \approx 0.08$$

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

# Quantum state orthogonalization: experiment

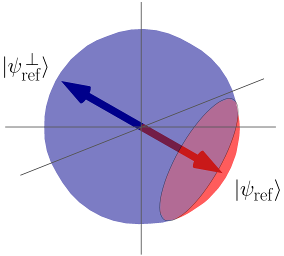



a) $\{z_1 = 0.2, z_2 = -0.2\}$ (b) $\{z_1 = 0.2, z_2 = -0.2 - 0.1i\}$ (c) $\{z_1 = 0.2e^{i\frac{\pi}{4}}, z_2 = -0.2e^{-i\frac{\pi}{4}}\}$

G. Zhu, O. Kálmán, K. Wang, L. Xiao, X. Zhan, Z. Bian, T. Kiss, P. Xue, PRA, in press

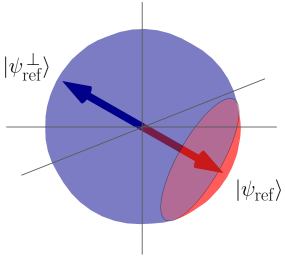
# Quantum state matching




- define a reference state: $|\psi_{\rm ref}\rangle$
- define a neighborhood: $\varepsilon = |\langle\psi \mid \psi_{\rm ref}\rangle|$
- find which $f$ corresponds to it
- find implementation of $f$
  - 2-qubit unitary+post-selection

O. Kálmán and T. Kiss, Phys. Rev. A **97**, 032125 (2018)

# Quantum state matching




- define a reference state: $|\psi_{\text{ref}}\rangle$
- define a neighborhood: $\varepsilon = |\langle\psi \mid \psi_{\text{ref}}\rangle|$
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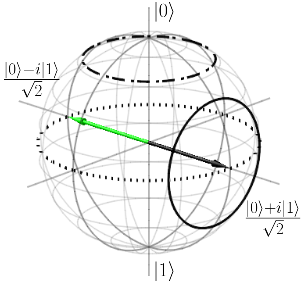


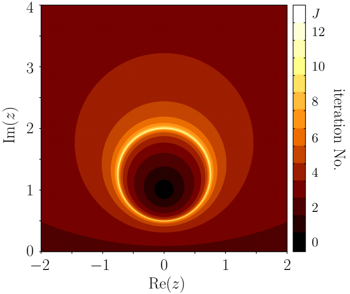


$$|\psi_{\text{ref}}\rangle = \frac{|0\rangle + i\,|1\rangle}{\sqrt{2}}$$

$$\varepsilon^2 = 0.9$$

O. Kálmán and T. Kiss, Phys. Rev. A **97**, 032125 (2018)

Thank you for your attention!

MAGYAR TUDOMÁNYOS AKADÉMIA 1825

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HUNQUTECH